APOLAR SYSTEMS OF BILINEAR FORMS

BY

JANIE CAMPBELL LAPSLEY

A.B., Agnes Scott College, 1934

M.S., University of Chicago, 1936

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
MATHEMATICS IN THE GRADUATE SCHOOL OF
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$$F \equiv (\alpha\alpha''\xi)(\alpha'\alpha'''\xi)(\beta\beta'\eta)(\beta''\beta'''\eta)(\gamma\gamma'\gamma''\gamma''')$$

$$K \equiv (\alpha'''\varkappa)(\beta\beta'''\eta)(\beta'y)(\beta''y)(\alpha\alpha'\alpha'')(\gamma\gamma'\gamma''\gamma'''), \quad (\eta y)=0.$$

These contain 36 and 45 constants respectively. In terms of the coefficients of $T(2,2,4)$ they are

$$F \equiv (a'\xi)(a''\xi)(b'''\eta)(b^{IV}\eta)(aa'''a'^{IV})(\delta\delta'\delta''\delta'''\delta'^{IV})(bb'b'') = 0$$

$$K \equiv (a'''a'^{IV}x)(b''\eta)(bb'''y)(b'b'^{IV}y)(aa'a'')(\delta\delta'\delta''\delta'''\delta'^{IV}), \ (\eta y)=0$$

To simplify notation these are written

$$F = (f\xi)^2(g\eta)^2 \tag{3}$$

$$K = (\kappa x)(l\eta)(\lambda y)^2, \quad (\eta y) = 0. \tag{4}$$

Since the roles of x and y [ξ and η] are similar, there is a combinant of the form

$$M = (\mu y)(n\xi)(\nu x)^2, \quad (\xi x) = 0. \tag{5}$$

This has 45 constants and so completes the list of fundamental combinants.

3. A Manifold of Dimension three and Order three.

It has been shown by A.B. Coble[1] that in $S_3(z)$ the locus of a point z, whose correlation

$$(\alpha x)(\beta y)(\gamma z) = 0$$

is degenerate, is a cubic surface. The singular points of the correlation, x and y in S_2 and S_2' respectively, are corresponding points under a Cremona transformation, T, of order five. T has six fundamental points, say p_i in S_2, q_i in S_2' $(i=1,\ldots,6)$.

Consider K=0,

$$(\kappa x)(l\eta)(\lambda y) = 0.$$

For a fixed x, this is the equation of a connex $C(2,1)$ in S_2'. The seven singular points of the connex (i.e., points such that every line on them is in the connex) are the six points q_i and the point y corresponding to x under T. To get the equations for T, let $x=x^{(0)}$ and $x=x^{(1)}$ and eliminate η between the equations

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(1) Cf. (2) in the bibliography.

$$(\kappa x^{(0)})(l\eta)(\lambda y)^2 = 0$$
$$(\kappa x^{(1)})(l\eta)(\lambda y)^2 = 0$$
$$(\eta y) = 0.$$

Letting $\xi = \overline{x^{(0)}x^{(1)}}$, this gives

(6) $$(\kappa\kappa'\xi)(ll'y)(\lambda y)^2(\lambda' y)^2 = 0.$$

Similarly the equation for the transformation from S_2 to S_2' is

(7) $$(\mu\mu'\eta)(nn'x)(\nu x)^2(\nu' x)^2 = 0.$$

The locus of a point u in $S_4(u)$ whose resulting correlation, $(a\xi)(b\eta)(\delta u) = 0$, is singular is a three dimensional manifold, say M, of order three,

(8) $$|a_i b_j (\delta u)| = 0, \quad i,j=0,1,2.$$

The correlations of $T(2,2,4)$ whose determinants are of rank one have the form $(\xi x^{(0)})\cdot(\eta y^{(0)}) = 0$; and by the apolarity condition $(\alpha x^{(0)})(\beta y^{(0)})(\gamma z) \equiv 0$ in z. The six pairs p_i, q_i are the only ones which satisfy this condition; therefore, there are exactly six correlations of rank one. The six points of M which give rise to these correlations are nodes and are denoted by n_i $(i=1,\ldots,6)$, respectively.

On a generic point of M there are exactly six lines which are contained in the manifold. These lines may be divided into three different systems. One system is the ∞^2 set given by the vanishing of a generic row of the determinant in equation (8); i.e., $(a\xi)b_j(\delta u) = 0$, $j=0,1,2$. Call the lines of this system J-lines and denote the line given by the fixed ration $\xi^{(\omega)}$ by $J(\xi^{(\omega)})$. Another system is the ∞^2 lines given by the vanishing of a generic column of the determinant, $a_i(b\eta)(\delta u) = 0$, $i=0,1,2$. Call these K-lines. These two systems have similar properties and their roles may be interchanged.

From the equation for the line $J(\xi^{(\omega)})$ it is evident that all correlations resulting from points on this line have $\xi^{(\omega)}$ as the singular line in S_2 and conversely. Thus there is a

(1,1) correspondence between the J-lines and the lines of S_2. The envelope of the singular lines in S_2' is the conic

$$(f\xi^{(0)})^2(g\eta)^2 = 0. \tag{9}$$

This conic is degenerate if and only if the lines $\xi^{(0)}$ is on one of the points p_i; therefore,

$$(f\xi)^2(f'\xi)^2(f''\xi)^2(gg'g'')^2 = 0$$

is the equation of the p_i.

Similarly, the correlations resulting from the points of $K(\eta^{(0)})$ have the same singular line in S_2', $\eta^{(0)}$; thus there is a (1,1) correspondence between the K-lines and the lines of S_2'. The singular lines in S_2 envelope the conic

$$(f\xi)^2(g\eta^{(0)})^2 = 0, \tag{10}$$

and the equation for the points q_i is

$$(ff'f'')^2(g\eta)^2(g'\eta)^2(g''\eta)^2 = 0.$$

In general two lines of the same system do not intersect; however, on a node there are ∞^1 J-lines [K-lines]. If $\lambda_0\xi^{(0)} + \lambda_1\xi^{(1)}$ is a pencil of lines on p_i, the J-lines on n_i are given by

$$(a\overline{\lambda_0\xi^{(0)} + \lambda_1\xi^{(1)}})b_j(\delta u) \equiv \lambda_0(a\xi^{(0)})b_j(\delta u) + \lambda_1(a\xi^{(1)})b_j(\delta u) = 0,\ j=0,1,2,$$

These J-lines form a cubic cone with vertex at n_i. The K-lines on n_i also form a cubic cone and the two cones have only the five lines $\overline{n_i n_j}$ $(j\neq i)$ in common. A hyperplane intersects each of the cones in a cubic curve. The two cubic curves have five points in common and consequently are on a quadric surface, one curve being [2,1] and the other [1,2].

Consider a line of M which is not a J-line or a K-line; call it L. By the proper transformation of coordinates the equation of this is brought into the form $u_2=u_3=u_4=0$. Moreover, assume that the determinant in (8) is written in such a way that the J- and K-lines on (1,0,0,0,0) make the first row and column, respectively, vanish and those on (0,1,0,0,0) make the second row and column vanish. Equation (8) then has the form

$$(8') \quad \begin{vmatrix} b_{00}(u_2,u_3,u_4) & b_{01}(u_2,u_3,u_4) & a_{02}u_1 + b_{02}(u_2,u_3,u_4) \\ b_{10}(u_2,u_3,u_4) & b_{11}(u_2,u_3,u_4) & a_{120}u_0 + b_{12}(u_2,u_3,u_4) \\ a_{201}u_1 + b_{20}(u_2,u_3,u_4) & a_{210}u_0 + b_{21}(u_2,u_3,u_4) & a_{22}(u_2,u_3) + b_{22}(u_2,u_3,u_4) \end{vmatrix} = 0$$

where the $b_{ij}(u_2,u_3,u_4)$ are linear functions of u_2,u_3,u_4, $a_{22}(u_0,u_1)$ is a linear function of u_0,u_1 and the a_{ijk} are constants. An examination of this shows that the points of L make every element of a second minor vanish. This property defines the third system of ∞^2 lines, namely those lines given by the vanishing of the elements of a second minor of the determinant in the equation of M. The lines of this system are called L-lines. On a generic point of M there are four L-lines, one J-line and one K-line.

The correlations of T(2,2,4) are given by bordering the determinant in (8) [or (8')] with x, x'; y, y'. The correlations resulting from the points of L, $u_2=u_3=u_4=0$, are

$$\begin{vmatrix} 0 & 0 & 0 & x_0 & x_0' \\ 0 & 0 & 0 & x_1 & x_1' \\ a_{201}u_1 & a_{210}u_0 & a_{22}(u_0,u_1) & x_2 & x_2' \\ y_0 & y_1 & y_2 & 0 & 0 \\ y_0' & y_1' & y_2' & 0 & 0 \end{vmatrix} = 0$$

or $$\eta_2\left[a_{021}u_1\xi_0 + a_{120}u_0\xi_1 + a_{22}(u_0,u_1)\xi_2\right] + \eta_1 a_{210}u_1\xi_2 + \eta_0 a_{201}u_1\xi_2 = 0$$

where $\xi = \overline{xx'}$, $\eta = \overline{yy'}$. The singular lines of these correlations are

$$\xi_0 : \xi_1 = -a_{120}u_0 : a_{021}u_1, \quad \xi_2 = 0, \quad \text{in } S_2,$$

and $$\eta_0 : \eta_1 = -a_{210}u_0 : a_{201}u_1, \quad \eta_2 = 0, \quad \text{in } S_2'.$$

In each plane the singular lines of the correlations resulting from the points of an L-line form a pencil and in each correlation the vertices of the pencils are corresponding points[1]. Moreover, the vertices are corresponding points in T. Conversely, every pair of co-points of T are the vertices for such

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(1) In the correlation having ξ and η as singular lines, points x on ξ and y on η are said to be corresponding if every line on x goes into y and conversely.

pencils of singular lines. Thus there is a (1,1) correspondence between the L-lines and the pairs of co-points (x,y) in T.

4. Contact Transformation.

Consider the lineal element $[x^{(o)}|\xi^{(o)}]$ in S_2, $x^{(o)}$ on $\xi^{(o)}$ a generic line of S_2. Let $y^{(o)}$ be the point corresponding to $x^{(o)}$ in T. On $y^{(o)}$ there are two lines of the envelope given in (9) and the apolarity condition shows that under the correlation of T(2,2,3) having $x^{(o)}$ and $y^{(o)}$ as singular points one of these lines corresponds to $\xi^{(o)}$, say $\eta^{(o)}$. Thus $[x^{(o)}|\xi^{(o)}]$ determines a point $u^{(o)}$ on M, the intersection of $J(\xi^{(o)})$ and $K(\eta^{(o)})$; in a similar way $[y^{(o)}|\eta^{(o)}]$ determines the same point $u^{(o)}$. Conversely, consider $u^{(o)}$, a generic point of M, and let the J- and K-lines on it be denoted by $J(\xi^{(o)})$ and $K(\eta^{(o)})$; then the resulting correlation is $(a\xi)(b\eta)(\delta u^{(o)}) \equiv (\xi r^{(1)})\cdot(\eta s^{(2)}) - (\xi r^{(2)})\cdot(\eta s^{(1)}) = 0$, where $r^{(1)}$, $r^{(2)}$ are on $\xi^{(o)}$, $s^{(1)}$, $s^{(2)}$ on $\eta^{(o)}$. It follows from the apolarity condition that there is exactly one degenerate correlation of T(2,2,3) having singular points on $\xi^{(o)}$ and $\eta^{(o)}$ and having $\xi^{(o)}$ and $\eta^{(o)}$ as corresponding lines; let the singular points be $x^{(o)}$, $y^{(o)}$. The $u^{(o)}$ uniquely determines $[x^{(o)}|\xi^{(o)}]$ in S_2 and $[y^{(o)}|\eta^{(o)}]$ in S_2'. Hence there is a (1,1) correspondence between the points $u^{(o)}$ on M and the lineal elements $[x^{(o)}|\xi^{(o)}]$ of S_2, also between $u^{(o)}$ and the lineal elements $[y^{(o)}|\eta^{(o)}]$ of S_2'. Thus there exists a contact transformation which carries the lineal element $[x^{(o)}|\xi^{(o)}]$ of S_2 into the lineal element $[y^{(o)}|\eta^{(o)}]$ of S_2'.

To find the locus of the point u corresponding to $[x^{(o)}|\xi]$ where $x^{(o)}$ is fixed and ξ varies, consider the correlation, C, of T(2,2,3) having $x^{(o)}$, $y^{(o)}$ as singular points. Let $\xi^{(a)} + \lambda\xi^{(b)}$ and $\eta^{(a)} + \lambda\eta^{(b)}$ be corresponding lines in this correlation. The apolarity condition requires that $J(\xi^{(a)} + \lambda\xi^{(b)})$ and $K(\eta^{(a)} + \lambda\eta^{(b)})$ have a point in common. The locus of this point is a rational quartic curve. Under C, $\overline{x^{(o)}p_i}$ and $\overline{y^{(o)}q_i}$ are corresponding lines; these are singular lines in the correlation resulting from n_i and so

the n_i are on the curve. Each pair of co-points of T, (x,y), gives rise to a rational quartic on the n_i, and through any point of M, not a node passes exactly one of the curves. M may be regarded as being generated by this system of rational quartic curves on the six points. Requiring that a rational quartic be on the six points n_i and that the n_i are given by the same binary sextic giving the lines $\overline{xp_i}$ [or $\overline{yq_i}$] in the pencil of lines on x [or y] gives this ∞^2 system of quartics.

To obtain the equation for the contact transformation between lineal elements in S_2 and S_2', consider $[x^{(\omega)}|\xi'^{(\omega)}]$ where both $x^{(\omega)}$ and $\xi'^{(\omega)}$ are fixed. The corresponding point is the co-point of $x^{(\omega)}$ in T and its equation is given by substituting $x=x'^{(\omega)}$ in (7). The equation of the corresponding line is obtained by substituting $x=x^{(\omega)}$, $\xi=\xi'^{(\omega)}$ in M=0 and it is

$$(\mu y)(n\xi'^{(\omega)})(\nu x'^{(\omega)})^2 = 0.$$

For $[y^{(\omega)}|\eta'^{(\omega)}]$ the corresponding point is given by substituting $y'^{(\omega)}$ in equation (6), and the corresponding line by substituting $y^{(\omega)}$, $\eta'^{(\omega)}$ in K=0.

This contact transformation can be used to describe further a property of T. Consider a fixed $\xi'^{(\omega)}$ in S_2; by T it is transformed into a quintic curve $C^{(5)}(\xi'^{(\omega)})$, and the line conic corresponding to $\xi'^{(\omega)}$, say $E(\xi'^{(\omega)})$, is $(f\xi'^{(\omega)})^2(g\eta)^2 = 0$. Under the contact transformation $[x|\xi'^{(\omega)}]$ corresponds to $[y|\eta]$ where y is on $C^{(5)}(\xi'^{(\omega)})$ and η is in $E(\xi'^{(\omega)})$. Therefore the rational $C^{(5)}(\xi'^{(\omega)})$ and $E(\xi'^{(\omega)})$ are perspective rational curves; i.e., point and line determined by like parameters are incident. A rational quintic has a unique perspective conic, so $C^{(5)}(\xi'^{(\omega)})$ determines $E(\xi'^{(\omega)})$. An investigation of all possible types of Cremona transformations of order five shows that only in the case of six fundamental points can the curves of the homaloidal net have non-degenerate perspective conics. Moreover, the entire system of apolar bilinear forms is determined by selecting the six points p_i in

S_2 (or the associated six points q_i in S_2') so the transformation T is a general Cremona transformation of order five having six F-points. Therefore, for a Cremona transformation of order five the curves of the homaloidal net and their perspective conics determine a contact transformation between the two planes.

5. Projections from Lines of M.

First consider h, a generic line of **M**, and the plane projection from h of the lines of **M** cutting h. Without any loss of generality all projections can be made upon the plane (say π) $u_0=u_1=0$, and u_2,u_3,u_4 can be considered as the coordinates in this plane. The projection from h of the lines cutting it is a quintic curve $C^{(5)}(h)$. The nodes n_i are projected from h into the points t_i on π and these points are on $C^{(5)}(h)$. The hyperplanes tangent to M at points on h cut π in lines whose envelope is a conic, E(h).

From a generic J-line, say $J(\xi'^{(0)})$, project the K-lines intersecting it upon π. The proper transformation of coordinates in S_4 takes the equation of $J(\xi'^{(0)})$ into the form $u_2=u_3=u_4=0$. The projection upon π of the K-line on the point $u^{(0)} \equiv (u_0^{(0)},u_1^{(0)},0,0,0)$ is

$$u_0 = u_1 = 0$$
$$u_i = f_i(u_0^{(0)},u_1^{(0)}), \quad i=2,3,4,$$

where the $f_i(u_0^{(0)},u_1^{(0)})$ are quintic functions of $u_0^{(0)},u_1^{(0)}$. As $u^{(0)}$ moves along $J(\xi'^{(0)})$, this point in π traces out a rational curve of order five, $C^{(5)}(\xi'^{(0)})$. Thus if the projection described above is made from a J-line [or a K-line] the $C^{(5)}(\xi'^{(0)})$ [or $C^{(5)}(\eta'^{(0)})$] is a rational curve. In S_2' on q_i there are two lines of the envelope (3); so in S_4 there are two K-lines on n_i which intersect $J(\xi'^{(0)})$. Therefore, the projections of the nodes, t_i, are double points on $C^{(5)}(\xi'^{(0)})$.

A point of $J(\xi'^{(0)})$ uniquely determines the $K(\eta'^{(0)})$ and the tangent hyperplane on it, hence the projection of these; therefore, in π there is a (1,1) correspondence between the points

of $C^{(5)}(\xi^{(\infty)})$ and the lines of the conic $E(\xi^{(\infty)})$. Moreover, corresponding point and line are incident. Hence, the rational $C^{(6)}(\xi^{(\infty)})$ and $E(\xi^{(\infty)})$ are perspective curves.

Consider the totality of curves $C^{(6)}(\xi)$ obtained by making similar projections from all J-lines. There is an ∞^2 system of $C^{(5)}(\xi)$ on the six points t_i. These curves form a homaloidal net and they are in (1,1) correspondence with the lines of S_2. This establishes a quintic Cremona transformation between S_2 and π. A further study of the correspondence shows that the F-points in S_2 are the p_i. Thus the quintic Cremona transformation between S_2 and π is of the same type and has the same F-points in S_2 as T; hence, there is a projective transformation between π and S_2' which takes the t_i into the q_i. This property is sufficient to define the J-lines; i.e., the J-lines may be thought of as those lines from which the six nodes of M are projected into a planar six-point projective to the q_i in S_2'. A similar discussion shows that the K-lines may be defined as those from which the six nodes are projected into a planar six-point projective to the p_i in S_2'.

The projection upon π, from a generic L-line of the J-lines intersecting it is a conic $C^{(2)}(L)$. The points t_i, the projections of the nodes of M, are on $C^{(2)}(L)$. Similarly, the projections of the K-lines intersecting the L-line is a conic which contains the six points t_i; therefore, the conics are identical. The line conic E(L) is perspective to this conic. This property may be used to define the L-lines. An L-line is a line from which the six nodes of M are projected into the six points on a conic, and the binary sextic giving the parameters of these six points is the same as the one giving the pencil of six lines $\overline{xp_i}$ on x [or $\overline{yq_i}$ on y, where y is the co-point of x in T].

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Vita

Janie Campbell Lapsley was born January 27, 1913, in Clarkton, North Carolina. Her early education was received in the public schools of Tarboro, N.C. and Columbia, S.C. and in 1929 she graduated from the Columbia High School. Her undergraduate work was done at Agnes Scott College and she received the degree of Bachelor of Arts in 1934. In 1936 she received the degree of Master of Science from the University of Chicago. The next three years she was an instructor in mathematics and chemistry at Mississippi Synodical College. Since then she has been a graduate student at the University of Illinois, holding an assistantship in the mathematics department since 1940.

She is a member of Phi Beta Kappa, the Society of Sigma Xi and the American Mathematical Society.